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Higher charges in dynamical spin chains for SYM theory

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Abstract: We construct, to the first two non-trivial orders, the next conserved charge in the $su(2|3)$ sector of $\mathcal{N} = 4$ Super Yang-Mills theory. This represents a test of integrability in a sector where the interactions change the number of sites of the chain. The expression for the charge is completely determined by the algebra and can be written in a diagrammatic form in terms of the interactions already present in the hamiltonian. It appears likely that this diagrammatic expression remains valid in the full theory and can be generalized to higher loops and higher charges thus helping in establishing complete integrability for these dynamical chains.

Keywords: Integrable Equations in Physics, Extended Supersymmetry, AdS-CFT and dS-CFT Correspondence, 1/N Expansion.

Contents

1. Introduction and summary

Recently, much effort has been devoted to the study of maximally supersymmetric ($\mathcal{N} = 4$) Super Yang-Mills theory (SYM) as a first step toward uncovering some of the mysteries shrouding strongly coupled gauge theories such as QCD. One very fruitful line of investigation, pioneered by the work of Minahan and Zarembo [1], is to employ the fact that the matrix of planar anomalous dimensions of local operators of this gauge theory can be though of as the hamiltonian of a quantum spin chain. In [1] the authors considered the scalar sector of the theory at first order in perturbation theory, but the result was quickly generalized (still at one loop) to the complete set of operators in [2]. The complete one loop dilatation operator of $\mathcal{N} = 4$ SYM is a $psu(2, 2|4)$ invariant spin chain, which turns out to be integrable.[1] Integrability introduces an extremely powerful tool in the arsenal of gauge theory computations, as it allows one to explicitly diagonalize the matrix of anomalous dimensions by the method of the Bethe ansatz. Although one does not yet have a complete understanding of the dilatation operator, by now the latter is known to rather high orders in perturbation theory in various closed sub-sectors of the gauge theory.

The particular sub-sector that will be analyzed in this paper is the so-called $su(2|3)$ sector [11], formed out of a two-component fermion and three scalars, which is known to be closed to all orders in perturbation theory. This sector inherits a global $su(2|3)(\in psu(2, 2|4))$ symmetry, and it has proved to be a useful testing ground for ideas related to higher loop integrability in the complete gauge theory. From the work of Beisert [11], the dilatation operator is known to the third order in perturbation theory in this sector.

Work has also been done on a $su(2)(\in su(2|3))$ sub-sector, where the corresponding three loop spin chain has been shown by Serban and Staudacher [12] to be embedded in a well known integrable long ranged spin chain, known as the Inozemtsev spin chain. Extending this work, the all loop Bethe equations and factorized scattering matrices have also been proposed in [13] for the $su(2)$ sub-sector of the gauge theory. The proposals for all loop Bethe ansatz and factorized S matrices have also been generalized to include

[1] Other aspects of integrability had already appeared in the context of QCD [3 – 5]. See [6 – 10] for further work along these lines.

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the $su(1|1), sl_2$, $su(1|2)$ and $psu(1,1|2)$ sub-sectors [14] (see also [15]). Assuming that the proposed S matrices and Bethe equations correctly describe all these sectors of the gauge theory, then, owing to the factorized nature of the S matrix, integrability would be obvious in these sectors as well. Finally, also in [14] a remarkable ansatz was proposed that generalizes all these results to the full $psu(2,2|4)$ algebra. So far, no corresponding factorized S matrix is known that can incorporate the full $psu(2,2|4)$ sector, hence, the Bethe equations proposed in [14] assume the existence of an underlying integrable spin chain. The full $psu(2,2|4)$ sector of the gauge theory includes $su(2|3)$ sector as well, within which the dilatation operator is known to the third order in perturbation theory. Since many of the novel features, such as the dynamical nature of the spin chains that are present in the complete gauge theory dilatation operator at higher loops are also present in the $su(2|3)$ sub-sector, it is important to make further checks of integrability for such closed sub-sectors.

Much of the intuition that goes into the construction of the all loop Bethe ansatz for the various sectors of the gauge theory comes from previous detailed studies of the $su(2)$ invariant long ranged spin chains [16 − 19]. However, when one ventures into the $su(2|3)$ sector, one encounters novel spin chains which do not seem to have been studied in the past. Specifically, since the 'length' of the spin chains (i.e. the number of SYM fields inside a single trace operator) is no longer a good quantum number, one has to deal with spin chains that do not preserve the number of sites. Such dynamical chains are not included in any of the smaller sectors mentioned above and it is important to check in what sense (if any) the notion of integrability generalizes to these chains.

For a given quantum spin chain, integrability is in general quite hard to establish. Ideally, one would like to establish integrability of a spin chain hamiltonian by relating it to a transfer matrix satisfying a Yang-Baxter algebra, or by showing that its scattering matrix is factorized. In all the cases for which this is possible, one can show that the hamiltonian is member of a family of mutually commuting conserved charges. Hence, given a spin chain that has a chance of being integrable, a test of its integrability [20] can be taken to be the existence of a higher conserved charge whose rank (in the sense of the range of interactions) is greater than that of the hamiltonian (see also [21, 22] for a σ-model perspective).

In the present context, the simplest way in which one can try to generalize this notion of integrability to higher loops is for the conserved charges $\mathcal{H}_s$ to be "deformed" by the coupling constant $g = g_{YM}\sqrt{N_c}$, (the square root of the 't Hooft coupling) as $\mathcal{H}_s \to \mathcal{H}_s(g)$ in such a way that the mutual commutators are still vanishing, at least when Taylor expanded around the origin [23]. This scenario has been referred to as 'perturbative integrability' in [11]. Precisely in [11] the hamiltonian for the $su(2|3)$ sector of the theory was computed to three loops and an argument for integrability, based on the degeneracy of the parity pairs, was given. The one loop $su(2|3)$ spin chain hamiltonian is of the type that has been studied in the condensed matter physics literature in the context of the Hubbard model [24], and it can be regarded as a supersymmetric generalization of the celebrated Heisenberg hamiltonian with nearest neighbor exchange interactions. The first non-trivial conserved charge of this one-loop hamiltonian (and indeed all the higher ones) are known in a very explicit form. Hence, a test of perturbative integrability in the $su(2|3)$ sector consists in constructing a g dependent deformation of the next conserved charge.

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In the present work we perform this test and construct such deformation to the first two non-trivial orders in g. This new conserved charge is completely determined and can be expressed in a very simple graphic form as a quadratic combination of terms in the hamiltonian that should easily generalize to higher loops, higher charges and, most importantly, different integrable systems, including the full $psu(2,2|4)$ algebra. In the light of discussions such as [20] this is a very strong check that integrability does indeed survive in the dynamical chain.

2. Test of perturbative integrability

"Perturbative Integrability" is the statement that for small values of the 't Hooft coupling $g = g_{YM}\sqrt{N_c}$ there exists an infinite[2] set of mutually commuting conserved charges $\mathcal{H}_s(g)$ (we will take $s \geq 2$) analytic at $g = 0$. Performing a Taylor expansion on the charges:

$$\mathcal{H}_s(g) = \sum_{k=0}^{\infty} \mathcal{H}_{s,k} g^k \tag{2.1}$$

we can expand the commutator of two charges to obtain an infinite set of relations

$$\sum_{k=0}^{l} [\mathcal{H}_{r,k}, \mathcal{H}_{s,l-k}] = 0\,. \tag{2.2}$$

The first relation obtained from (2.2) for $l = 0$ is the usual relation discussed in the context of integrable spin chains:

$$[\mathcal{H}_{r,0}, \mathcal{H}_{s,0}] = 0\,. \tag{2.3}$$

The lowest charge $\mathcal{H}_2(g)$ corresponds to the dilatation operator, to be thought of as the "hamiltonian" of the spin system. More specifically, adopting the standard notation, we write the total scaling dimension $\mathcal{D}$ of an operator as

$$\mathcal{D} = \mathcal{D}_0 + g^2 \mathcal{H}_2(g)\,, \tag{2.4}$$

where $\mathcal{D}_0$ is the classical (mass) dimension and an overall factor of g^2 has been extracted form the charge. Thus, with this conventions, $\mathcal{H}_2(0) = \mathcal{H}_{2,0}$ corresponds to the one loop matrix of anomalous dimensions. Although the Taylor expansion of $\mathcal{H}_2(g)$ may contain odd powers of g, it is possible to check that the eigenvalues of $\mathcal{D}$ always contain even powers of g.

As mentioned in the introduction, it is crucial to investigate sectors where the length of the chain is allowed to vary to test the validity of (2.2) in the context of $\mathcal{N} = 4$ SYM theory. The simplest such sector, closed to all loops, is the so-called $su(2|3)$ sector, consisting of three bosons Φ^a ($a = 1, 2, 3$) and two fermions Ψ^α ($\alpha = 1, 2$). For such sector, the hamiltonian $\mathcal{H}_2(g)$ is known to order $\mathcal{O}(g^4)$ from the work of Beisert [11] and a natural question is to try to construct the next charge $\mathcal{H}_3(g)$ to some order. To zeroth order the charge $\mathcal{H}_{3,0} = \mathcal{H}_3(0)$ is already known to exist because the system is nothing but the usual integrable system with hamiltonian described by (graded) permutations of nearest

[2] Or, for a chain of finite length L, of order L.

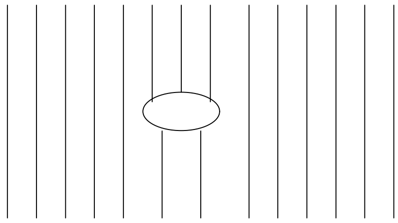

Figure 1: Graphical representation of an interaction that does not conserve the length of the chain. Each line ends on one site of a chain. The interaction, denoted by a blob, is supposed to be moved (summed) over the chain. For $\mathcal{N}=4$ SYM one considers only periodic chains, i.e. identifies the ends of the diagram.

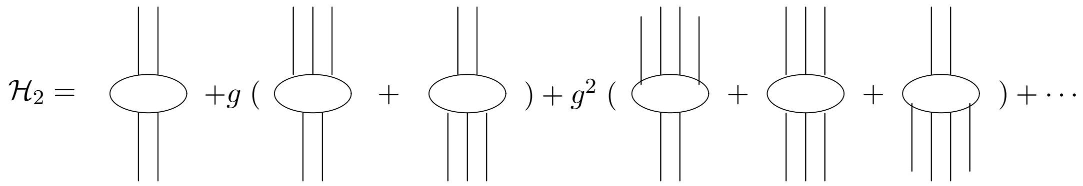

Figure 2: First conserved charge $\mathcal{H}_2(g)$ (hamiltonian). The fourth and sixth blob are identically zero in the $su(2|3)$ sector.

neighbors and the higher charges at $g=0$ are easily extracted from the transfer matrix. In this note we present the explicit expression for the first non-trivial term in the deformation $\mathcal{H}_{3,1}$. (See appendix for $\mathcal{H}_{3,2}$.) Such term already mixes chains of different lengths and thus represent a novel test of integrability.

In presenting the explicit expressions for these charges, we will use Beisert's notation [11] to denote the local interactions that, when summed over the chain, give rise to the charges. Briefly, an arbitrary $su(2|3)$ spin chain can be written as $\mathrm{tr}(W^{A_1}\dots W^{A_n})$, where $W^A=\Phi^a$ or Ψ^α, that is, we use capital letters to collectively describe all five spin states, while lower case Latin letters and Greek letters describe the three Bosonic and the two fermionic values of the spins respectively. A generic local interaction can be written as

$$\begin{Bmatrix} A_1\dots A_p \\ B_1\dots B_q \end{Bmatrix} \tag{2.5}$$

and can be thought of as a "machine" that runs along the chain and whenever it finds the combination of indices $A_1\dots A_p$ it replaces it with the combination $B_1\dots B_q$. In a "dynamical" chain the number of sites need not be conserved, i.e. $p\neq q$ (figure 1).

In the following, we will only need to know the expression for $\mathcal{H}_{2,0}$ and $\mathcal{H}_{2,1}$. (The latter, together with $\mathcal{H}_{2,2}$, is used to determine the two loop anomalous dimension.) We will also need the expressions for the supercharge $\mathcal{Q}^a_\alpha(g)$ to the same order. (The other supercharge, usually denoted by $\mathcal{S}^\alpha_a(g)$ will not give further constraints on $\mathcal{H}_3(g)$.) The diagrammatic expression for $\mathcal{H}_2(g)$ and $\mathcal{H}_3(g)$ are given in figure 2 and figure 3 respectively

Thus, we take as our hamiltonian, to zeroth and first order [11]:

$$\mathcal{H}_{2,0} = \begin{Bmatrix} AB \\ AB \end{Bmatrix} - (-1)^{(A)(B)} \begin{Bmatrix} AB \\ BA \end{Bmatrix}$$

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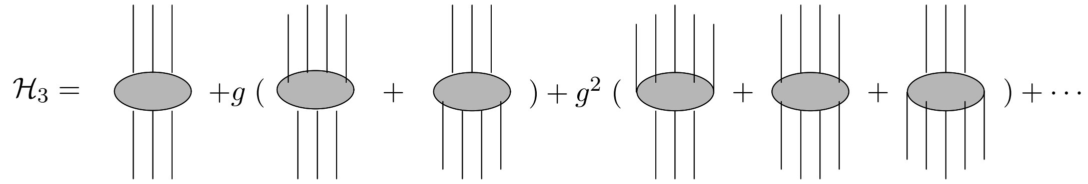

Figure 3: Second conserved charge $\mathcal{H}_3(g)$. The fourth and sixth blob are identically zero in the $su(2|3)$ sector.

$$
\begin{aligned}
&= \begin{Bmatrix} ab \\ ab \end{Bmatrix} + \begin{Bmatrix} \alpha b \\ \alpha b \end{Bmatrix} + \begin{Bmatrix} a\beta \\ a\beta \end{Bmatrix} + \begin{Bmatrix} \alpha\beta \\ \alpha\beta \end{Bmatrix} - \begin{Bmatrix} ab \\ ba \end{Bmatrix} - \begin{Bmatrix} \alpha b \\ b\alpha \end{Bmatrix} - \begin{Bmatrix} a\beta \\ \beta a \end{Bmatrix} + \begin{Bmatrix} \alpha\beta \\ \beta\alpha \end{Bmatrix}, \\
\mathcal{H}_{2,1} &= \epsilon_{abc}\epsilon_{\alpha\beta} \begin{Bmatrix} abc \\ \alpha\beta \end{Bmatrix} + \epsilon_{abc}\epsilon_{\alpha\beta} \begin{Bmatrix} \alpha\beta \\ abc \end{Bmatrix}. \qquad (2.6)
\end{aligned}
$$

We used the freedom allowed by a scaling of the coupling constant and a phase rotation to fix all coefficients in (2.6) without loss of generality. (In particular, we dropped a factor of $-1/\sqrt{2}$ in $\mathcal{H}_{2,1}$ that is required to get the right values for the anomalous dimensions but is irrelevant for our purpose.) The symbol (A) (and more generally $(A_1 \ldots A_n)$) at the exponent is equal to zero if the quantity within parenthesis is bosonic and equal to one if it is fermionic, thus enforcing the grading of the permutation.

In the same spirit, we define the supercharges

$$
\mathcal{Q}^a_\alpha(g) = \sum_{k=0}^{\infty} \mathcal{Q}_{k}{}^{a}_{\alpha}\, g^k\,, \qquad (2.7)
$$

where

$$
\mathcal{Q}_{0}{}^{a}_{\alpha} = \begin{Bmatrix} a \\ \alpha \end{Bmatrix} \qquad \text{and} \qquad \mathcal{Q}_{1}{}^{a}_{\alpha} = \epsilon_{abc}\epsilon_{\alpha\beta} \begin{Bmatrix} \beta \\ bc \end{Bmatrix}. \qquad (2.8)
$$

For the third charge, to zeroth order we have:

$$
\mathcal{H}_{3,0} = (-1)^{(C)(AB)} \begin{Bmatrix} ABC \\ CAB \end{Bmatrix} - (-1)^{(A)(BC)} \begin{Bmatrix} ABC \\ BCA \end{Bmatrix} \qquad (2.9)
$$

which when written out in detail reads as:

$$
\begin{aligned}
\mathcal{H}_{3,0} = &\begin{Bmatrix} abc \\ cab \end{Bmatrix} - \begin{Bmatrix} abc \\ bca \end{Bmatrix} + \begin{Bmatrix} ab\gamma \\ \gamma ab \end{Bmatrix} - \begin{Bmatrix} ab\gamma \\ b\gamma a \end{Bmatrix} + \begin{Bmatrix} a\beta c \\ ca\beta \end{Bmatrix} - \begin{Bmatrix} a\beta c \\ \beta ca \end{Bmatrix} + \\
&+ \begin{Bmatrix} \alpha bc \\ c\alpha b \end{Bmatrix} - \begin{Bmatrix} \alpha bc \\ bc\alpha \end{Bmatrix} - \begin{Bmatrix} a\beta\gamma \\ \gamma a\beta \end{Bmatrix} - \begin{Bmatrix} a\beta\gamma \\ \beta\gamma a \end{Bmatrix} - \begin{Bmatrix} \alpha b\gamma \\ \gamma\alpha b \end{Bmatrix} + \begin{Bmatrix} \alpha b\gamma \\ b\gamma\alpha \end{Bmatrix} + \\
&+ \begin{Bmatrix} \alpha\beta c \\ c\alpha\beta \end{Bmatrix} + \begin{Bmatrix} \alpha\beta c \\ \beta c\alpha \end{Bmatrix} + \begin{Bmatrix} \alpha\beta\gamma \\ \gamma\alpha\beta \end{Bmatrix} - \begin{Bmatrix} \alpha\beta\gamma \\ \beta\gamma\alpha \end{Bmatrix}. \qquad (2.10)
\end{aligned}
$$

This charge is obtained by commuting two graded permutations acting on three adjacent sites as shown in figure 4. As mentioned before, it is well known and easy to check that

$$
[\mathcal{H}_{2,0}, \mathcal{H}_{3,0}] = 0\,. \qquad (2.11)
$$

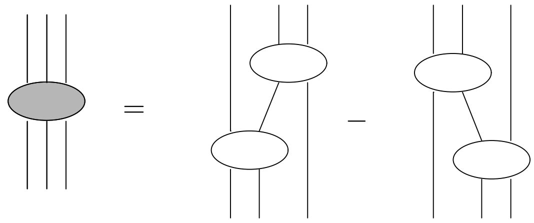

Figure 4: Diagrammatic representation of $\mathcal{H}_{3,0}$ in terms of $\mathcal{H}_{2,0}$.

We now proceed to construct the next term $\mathcal{H}_{3,1}$ such that relation (2.2) is satisfied for $l = 1$, namely:

$$[\mathcal{H}_{2,0}, \mathcal{H}_{3,1}] + [\mathcal{H}_{2,1}, \mathcal{H}_{3,0}] = 0\,, \tag{2.12}$$

and such that $\mathcal{H}_3(g)$ also commutes with the supercharges to this order:

$$[\mathcal{Q}_0{}^a_\alpha, \mathcal{H}_{3,1}] + [\mathcal{Q}_1{}^a_\alpha, \mathcal{H}_{3,0}] = 0\,, \tag{2.13}$$

This is nothing but a 'brute force' computation consisting in first writing the most general expression for the charge $\mathcal{H}_{3,1}$ and then fixing the coefficients in such a way that (2.12) and (2.13) are satisfied. The reason why we must also check (2.13) is that in this formulation, only the $su(2) \times su(3)$ bosonic subgroup of $su(2|3)$ is manifestly realized while the remaining generators will in general receive corrections, as is well known for the case of $\mathcal{H}_2(g)$ [11].

To obtain the most general expression for $\mathcal{H}_{3,1}$ we need to impose the restrictions coming from *manifest* $su(2) \times su(3)$ invariance and parity. The first implies that the charge can be constructed out of the following interactions (and their hermitian conjugates):

$$\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} Iabc \\ I\alpha\beta \end{Bmatrix}, \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} aIbc \\ I\alpha\beta \end{Bmatrix}, \ldots \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abcI \\ \alpha\beta I \end{Bmatrix}, \tag{2.14}$$

where the index I denotes either a bosonic index or a fermionic one and can be positioned anywhere in the upper or lower row.

However, since $\mathcal{H}_3(g)$ is parity odd, we can only allow combinations that are odd under the parity operation defined as:

$$\mathrm{P}\begin{Bmatrix} A_1 \ldots A_n \\ B_1 \ldots B_m \end{Bmatrix}\mathrm{P}^{-1} = (-1)^{m+n+i(i-1)/2+j(j-1)/2}\begin{Bmatrix} A_n \ldots A_1 \\ B_m \ldots B_1 \end{Bmatrix} \tag{2.15}$$

where i and j are respectively the number of fermionic terms in the upper and lower rows. This restricts the form of the charge to only ten unknown coefficients $A_1, A_2 \cdots E_1, E_2$. The most general ansatz for $\mathcal{H}_{3,1}$ can thus be written as:

$$\begin{aligned}\mathcal{H}_{3,1} = {}& A_1\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} kabc \\ \alpha k\beta \end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abck \\ \alpha k\beta \end{Bmatrix}\right) + \\ &+A_2\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} \gamma abc \\ \alpha\gamma\beta \end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abc\gamma \\ \alpha\gamma\beta \end{Bmatrix}\right) +\end{aligned}$$

$$
\begin{aligned}
&+B_1\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} kabc \\ \alpha\beta k\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abck \\ k\alpha\beta\end{Bmatrix}\right) + \\
&+B_2\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} \gamma abc \\ \alpha\beta\gamma\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abc\gamma \\ \gamma\alpha\beta\end{Bmatrix}\right) + \\
&+C_1\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} akbc \\ k\alpha\beta\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abkc \\ \alpha\beta k\end{Bmatrix}\right) + \\
&+C_2\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} a\gamma bc \\ \gamma\alpha\beta\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} ab\gamma c \\ \alpha\beta\gamma\end{Bmatrix}\right) + \\
&+D_1\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} akbc \\ \alpha k\beta\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abkc \\ \alpha k\beta\end{Bmatrix}\right) + \\
&+D_2\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} a\gamma bc \\ \alpha\gamma\beta\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} ab\gamma c \\ \alpha\gamma\beta\end{Bmatrix}\right) + \\
&+E_1\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} akbc \\ \alpha\beta k\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abkc \\ k\alpha\beta\end{Bmatrix}\right) + \\
&+E_2\left(\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} a\gamma bc \\ \alpha\beta\gamma\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} ab\gamma c \\ \gamma\alpha\beta\end{Bmatrix}\right) + \text{h.c.}
\end{aligned}
\tag{2.16}
$$

Further possible combinations:

$$
\epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} Iabc \\ I\alpha\beta\end{Bmatrix} - \epsilon_{abc}\epsilon_{\alpha\beta}\begin{Bmatrix} abcI \\ \alpha\beta I\end{Bmatrix} \tag{2.17}
$$

vanish identically on graded cyclic chains.

To fix the coefficients it is important to note that not all the ten terms in (2.16) are independent but that there are relations among them due to the $su(2)\times su(3)$ invariance cutting down on the number of independent coefficients. The first relation that one can utilize is that, for any I, J and K

$$
\epsilon_{abc}\left(\begin{Bmatrix} kabc \\ IJK\end{Bmatrix} - \begin{Bmatrix} abck \\ IJK\end{Bmatrix} - \begin{Bmatrix} akbc \\ IJK\end{Bmatrix} + \begin{Bmatrix} abkc \\ IJK\end{Bmatrix}\right) = 0\,. \tag{2.18}
$$

This relation is specific to $su(3)$ and can be proved by looking at the action of the l.h.s. on any single trace state. On a given single trace state, the l.h.s. will give a non-zero contribution only on parts of the state that contain four consecutive bosonic spins. Furthermore, because of the epsilon tensor, three of those spins have to be different. It is now straightforward to convince oneself that on any such 'block' of four bosonic spins the l.h.s. always sums up to zero. This relation immediately implies that D_1 and B_1 can be eliminated or, in other words, only the combinations A_1+D_1, B_1+C_1 and B_1+E_1 appear in the ansatz for $\mathcal{H}_{3,1}$. Acting on bosonic states with (2.12) then immediately fixes $B_1+E_1=0$, $A_1+D_1=-1$ and $B_1+C_1=1$.

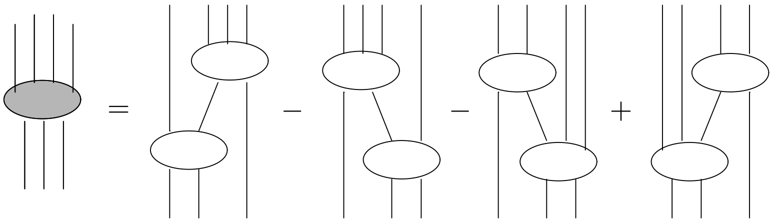

Figure 5: Diagrammatic representation of $\mathcal{H}_{3,1}$ in terms of $\mathcal{H}_{2,0}$ and $\mathcal{H}_{2,1}$. Each term on the r.h.s. corresponds to a pair of terms (bosonic and fermionic) in eq. (2.20). In principle, there could be two more diagrams like the last two but with $\mathcal{H}_{2,0}$ and $\mathcal{H}_{2,1}$ sharing two legs instead of one. These two diagrams cancel out in this case.

There is a similar relation, this time specific to $su(2)$, that allows one to eliminate some of the coefficients in the fermionic sector. Namely, for any I, J, K and L

$$\epsilon_{\alpha\beta}\left(\begin{Bmatrix} IJKL \\ \gamma\alpha\beta \end{Bmatrix} + \begin{Bmatrix} IJKL \\ \alpha\beta\gamma \end{Bmatrix} - \begin{Bmatrix} IJKL \\ \alpha\gamma\beta \end{Bmatrix}\right) = 0\,. \tag{2.19}$$

This relation implies that the term multiplying D_2 can be eliminated and the coefficients C_2 and E_2 change to $C_2 + D_2$ and $E_2 + D_2$ respectively.

After that, requiring (2.12) to be satisfied on fermionic states, we obtain $A_2 = 1$, $B_2 = 0$, $C_2 + D_2 = 1$ and $E_2 + D_2 = 0$, which fixes *all* the coefficients. The final formula for $\mathcal{H}_{3,1}$ thus takes the form:

$$\begin{aligned}\mathcal{H}_{3,1} = \epsilon_{abc}\epsilon_{\alpha\beta}&\left(-\begin{Bmatrix} kabc \\ \alpha k\beta \end{Bmatrix} + \begin{Bmatrix} abck \\ \alpha k\beta \end{Bmatrix} + \begin{Bmatrix} akbc \\ k\alpha\beta \end{Bmatrix} - \begin{Bmatrix} abkc \\ \alpha\beta k \end{Bmatrix}\right) + \\ +\epsilon_{abc}\epsilon_{\alpha\beta}&\left(+\begin{Bmatrix} \gamma abc \\ \alpha\gamma\beta \end{Bmatrix} - \begin{Bmatrix} abc\gamma \\ \alpha\gamma\beta \end{Bmatrix} + \begin{Bmatrix} a\gamma bc \\ \gamma\alpha\beta \end{Bmatrix} - \begin{Bmatrix} ab\gamma c \\ \alpha\beta\gamma \end{Bmatrix}\right) + \text{h.c.}\end{aligned} \tag{2.20}$$

It can also be checked that (2.13) does not introduce any further constraint, i.e. supersymmetry is preserved as well.

3. Concluding remarks

Eq. (2.20) is suggestive because it shows that $\mathcal{H}_{3,1}$ can be expressed in terms of the interactions already present in the hamiltonian by a rather simple generalization of figure 4 shown diagrammatically in figure 5.

Having a diagrammatic approach at hand, it is conceivable that one may be able to utilize the intuition generated by the present work and construct the complete monodromy matrix, which generates all the conserved quantities for this spin chain. Such a construction could then be used to derive the Bethe equations for the dynamical spin chain and establish complete integrability for this many body problem. By some preliminary analysis, we are quite convinced that such simple structure generalizes to higher loops and higher charges.[3]

[3]After the first version of this paper was submitted to the `arXive` we have completed the construction of $\mathcal{H}_{3,2}$, see appendix.

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This is yet another encouraging sign that it may be possible to find a transfer matrix even for dynamical spin chains of this type.

One further interesting fact is that diagrammatic relations such as figure 4 and figure 5 do not know about the details of the underlying model and can in principle be applied to the full $psu(2,2|4)$ chain perhaps even allowing one to fix the coefficients of the dilatation operator to higher loops.

Apart from local conserved charges, complete integrability would also require an understanding of how the Yangian symmetry of the gauge theory dilatation operator may be realized in the context of the dynamical spin chains. This symmetry is known to be present, at least to the first few orders in perturbation theory, in many of the sub-sectors of the gauge theory that do not exhibit the dynamical behavior [25 − 30], and it is also known to be present in the full gauge theory dilatation operator at one loop [31]. Hence, a novel realization of Yangian symmetry, relevant for dynamical spin chains, remains a crucial piece of the puzzle to be discovered.

Finally, even for theories with less on no supersymmetries, it may be interesting to try deforming the chiral sector discussed in [7, 10] in a way that preserves integrability. This sector is known not to be closed beyond one loop but, given the fact that the antichiral 'impurities' appear with a mass gap in long operators it might be possible to develop a truncation scheme in which integrability generalizes to higher loops even in these cases.

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A. The complete two loop charge

In this appendix we simply present the diagrammatic expression (figure 6) of $\mathcal{H}_{3,2}$ in terms of $\mathcal{H}_{2,0}$, $\mathcal{H}_{2,1}$ and $\mathcal{H}_{2,2}$. In general, it should be possible to obtain the k-th loop contribution $\mathcal{H}_{3,k}$ in terms of bilinears $\mathcal{H}_{2,i}\mathcal{H}_{2,k-i}$ for $i = 0 \ldots k$.

Imposing that the integrability condition is obeyed by $\mathcal{H}_{3,2}$ fixes the values of the undetermined parameter δ_2 in [11]. (δ_1 is still undetermined and δ_3 is not a true free parameter since it multiplies a quantity that is identically zero.) With the normalizations as in [11], ($\alpha_1 = 1$ and $\alpha_3 = 0$), setting the rotation angles $\gamma_i = 0$ we find $\delta_2 = -5/4$.

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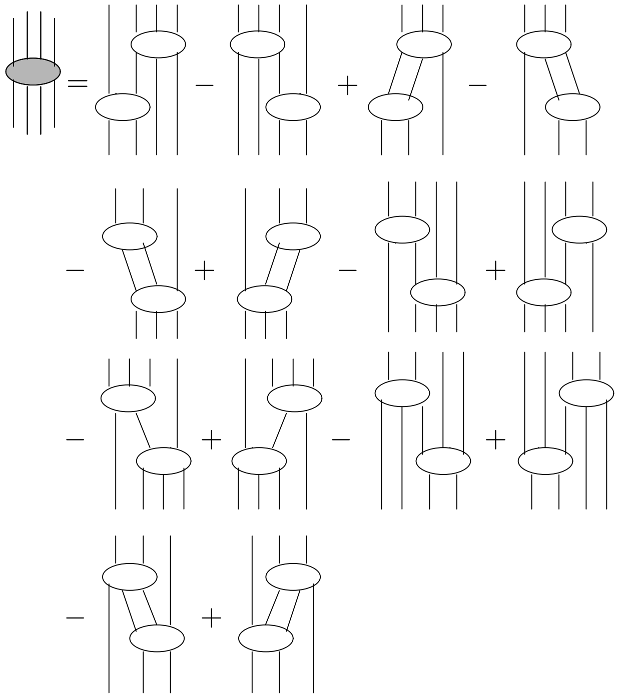

Figure 6: Diagrammatic representation of $\mathcal{H}_{3,2}$ in terms of $\mathcal{H}_{2,0}$, $\mathcal{H}_{2,1}$ and $\mathcal{H}_{2,2}$. Diagrams with less incoming and outgoing legs should be interpreted as having an extra spectator leg.

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